

# Solution of the Zabolotskaya-Khokholov Equation by Laplace Decomposition Method

Tahira Sumbal Shaikh<sup>1\*</sup>, Nauman Ahmed<sup>2</sup>, Naveed Shahid<sup>3</sup>, Zafar Iqbal<sup>4</sup>

<sup>1\*</sup>Department of Mathematics, Lahore College for Women University, Lahore, Pakistan

<sup>2,3,4</sup>Department of Mathematics & Statistics, University of Lahore, Lahore, Pakistan

\*Corresponding Author Email: [tahira.sumbal@lcwu.edu.pk](mailto:tahira.sumbal@lcwu.edu.pk)

Co-authors Email: [nauman.ahmed@math.uol.edu.pk](mailto:nauman.ahmed@math.uol.edu.pk), [naveed.shahid@math.uol.edu.pk](mailto:naveed.shahid@math.uol.edu.pk), [zafar.iqbal@math.uol.edu.pk](mailto:zafar.iqbal@math.uol.edu.pk)

**Abstract-**In this article, the Laplace Decomposition Method (LDM) is applied to earn semi analytical solution of the Zabolotskaya-Khokholove equation. It is manifested from the results that the proposed method leads to converge quickly the sequence of iterative solutions and the result approaches very close to the exact solution after less number of iterations as compared to the methods in existing literature.

**Keywords:** Approximate solution, Adomian Decomposition Method, Laplace Decomposition Method, Zabolotskaya-Khokholove equation.

## 1. Introduction

Mostly, in real world, we come across the differential equations which are most relevant tools to interpret and handle the real life problems. The differential equations also play a key role in different fields of physical sciences, engineering, fluid mechanics and more other fields. In preceding years, most of the mathematicians [1-22] have demonstrated in the field of differential equations to get their analytical and numerical solutions. Adomian introduced a technique named Adomian Decomposition Method [4, 5] to solve the various types of ordinary and partial differential

equations. This method helps us to solve the variety of linear and nonlinear differential equations [9-16]. On the basis of this method, Suheil [6, 7] established a new technique to solve the ordinary and partial differential equations named Laplace Adomian Decomposition Method. Most of the authors have used this method due to its significance. As it comprises two powerful techniques which provide us with the exact solutions in most of the cases [17-22].

Zabolotskaya-Khokholove(Z-K)equation is one dimensional nonlinear partial differential equation and is defined as:

$$u_{xt} = uu_{xx} + \alpha u_x^2, \quad 0 < x < 1, \quad t > 0 \tag{1.1}$$

This equation is of fundamental importance in nonlinear acoustic and nonlinear wave theory as well as it plays an important role in nonlinear acoustic and nonlinear wave theory. Two Russian mathematicians R. V. Khokhlove and E. A. Zabolotskaya [22] derived this equation about forty years ago and find the approximate solution. They discovered the characteristics of nonlinear wave beams. Currently Vibrant research is being done after the development of new medical instruments. Acoustic waves may be studied by the Z-K equation [1-3] in a nonlinear medium. The Z-K equation provides information about the propagation of sound beam or confined wave beam in nonlinear medium and studies the beam deformation. Researchers are investigating the new properties of nonlinear wave beams. The exact solution of the Z-K equation is found by many researchers with the help of different techniques. Saeed and Kalim [1] used the different techniques to have the solution of Z-K equation such as Adomian Decomposition Method, Variational Iteration Method and Homotopy Analysis Method. We will use the Laplace Decomposition Method for the semi analytical solution of Z-K equation as infinite series. This solution will help in the fields of physics, mathematics and engineering, to find the acoustic pressure.

## 2. Numerical Method

Consider the initial and boundary conditions for (1.1)

$$u_x(x, 0) = f(x) \text{ and } u(0, t) = g(t) \tag{2.1}$$

we consider

$$L(u(x, t)) = N(u(x, t)) \tag{2.2}$$

where  $L = \frac{\partial^2}{\partial x \partial t}$  and  $N$  represents the nonlinear operator.

By applying Laplace transformation on (2.2), we get

$$\mathcal{L}[L(u(x, t))] = \mathcal{L}[N(u(x, t))] \tag{2.3}$$

$$s\mathcal{L}[u_x(x, t)] - u_x(x, 0) = \mathcal{L}[N(u(x, t))] \tag{2.4}$$

$$s\mathcal{L}[u_x(x, t)] - f(x) = \mathcal{L}[N(u(x, t))] \tag{2.5}$$

$$\mathcal{L}[u_x(x, t)] = \frac{1}{s}f(x) - \frac{1}{s}\mathcal{L}[N(u(x, t))] \tag{2.6}$$

In next step, we will use Laplace Decomposition Method for the series solution such as,

$$u = \sum_{n=0}^{\infty} u_n(x, t) \text{ and } N[u(x, t)] = \sum_{n=0}^{\infty} P_n(u)$$

where  $P_n(u)$  represents Adomian polynomial of  $u_k$  where  $k = 0, 1, 2, \dots, n$ .

To find  $P_n$ , we use the formula,

$$P_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{k=0}^{\infty} \lambda^k u_k)]_{\lambda=0}, \quad n = 0, 1, 2, \dots \tag{2.5}$$

$$\mathcal{L}[\sum_{n=0}^{\infty} u_{n_x}(x, t)] = \frac{1}{s}f(x) - \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} P_n(u)] \tag{2.6}$$

or

$$[\sum_{n=0}^{\infty} \mathcal{L}(u_{n_x}(x, t))] = \frac{1}{s}f(x) - \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} P_n(u)] \tag{2.7}$$

when we compare the above equation, we get,

$$\mathcal{L}[(u_{0_x}(x, t))] = \frac{1}{s} f(x) \tag{2.8}$$

$$\mathcal{L}[(u_{1_x}(x, t))] = -\frac{1}{s} \mathcal{L}[P_0(u)] \tag{2.9}$$

$$\mathcal{L}[(u_{2_x}(x, t))] = -\frac{1}{s} \mathcal{L}[P_1(u)] \tag{2.10}$$

Thus the recursive relative, in general form is

$$\mathcal{L}[(u_{(n+1)_x}(x, t))] = -\frac{1}{s} \mathcal{L}[P_n(u)] \tag{2.11}$$

Now applying inverse Laplace transform to above equations

$$(u_{0_x}(x, t)) = K(x, t) \tag{2.12}$$

$$(u_{(n+1)_x}(x, t)) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L}(P_n(u)) \right] \tag{2.13}$$

where  $K(x, t)$  represented the expression with initial condition. First we applied Laplace transform on the right hand side of equation (2.13) then apply inverse Laplace transform and finally to get  $u(x, t)$ , integrate with respect to  $x$ .

Now we will illustrate the above procedure with numerical example.

### 3. Application

Consider Z-K equation (1.1)

$$u_{xt} = uu_{xx} + u_x^2, \quad 0 < x < 1, \quad t > 0 \tag{3.1}$$

with

$$f(x) = 2x, \quad g(t) = 0, \quad \alpha = 1. \tag{3.2}$$

$$\mathcal{L}[u_{xt}] = \mathcal{L}(N[u]) \tag{3.3}$$

where

$$N[u] = uu_{xx} + u_x^2 \tag{3.4}$$

$$s\mathcal{L}[u_x(x, t)] - u_x(x, 0) = \mathcal{L}[N(u(x, t))] \tag{3.5}$$

$$s\mathcal{L}[u_x(x, t)] - 2x = \mathcal{L}[N(u(x, t))] \tag{3.6}$$

$$s\mathcal{L}[u_x(x, t)] = \frac{1}{s}(2x) + \frac{1}{s} \mathcal{L}[N(u(x, t))] \tag{3.7}$$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \tag{3.8}$$

$$\left[ \sum_{n=0}^{\infty} \mathcal{L}(u_{n_x}(x, t)) \right] = \frac{1}{s}(2x) - \frac{1}{s} \mathcal{L}[\sum_{n=0}^{\infty} P_n(u)] \tag{3.9}$$

$$P_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{k=0}^{\infty} \lambda^k u_k)]_{\lambda=0}, \quad n = 0, 1, 2, \dots \tag{3.10}$$

$$\left[ \sum_{n=0}^{\infty} \mathcal{L}(u_{n_x}(x, t)) \right] = \frac{1}{s}(2x) - \frac{1}{s} \mathcal{L}[\sum_{n=0}^{\infty} P_n(u)] \tag{3.11}$$

Now

$$P_0 = N(u_0) = u_0 u_{0_{xx}} + u_{0_x}^2 \tag{3.12}$$

By comparing (3.9)

$$\mathcal{L}[(u_{0_x}(x, t))] = \frac{1}{s}(2x) \tag{3.13}$$

$$u_{0_x}(x, t) = (2x) \mathcal{L}^{-1} \left( \frac{1}{s} \right) \tag{3.14}$$

$$u_{0_x}(x, t) = (2x)(1) \tag{3.15}$$

$$u_0(x, t) = \int_0^x 2x dx = x^2 \tag{3.16}$$

$$u_0(x, t) = x^2 \tag{3.17}$$

$$\mathcal{L}[(u_{1_x}(x, t))] = \frac{1}{s} \mathcal{L}(P_0(u)) \tag{3.18}$$

$$P_0(u) = u_0 u_{0_{xx}} + u_{0_x}^2 = x^2(2) + 4x^2 = 6x^2 \tag{3.19}$$

$$\mathcal{L}[(u_{1_x}(x, t))] = \frac{1}{s} \mathcal{L}(6x^2) \tag{3.20}$$

$$\mathcal{L}[(u_{1_x}(x, t))] = \frac{6x^2}{s} \mathcal{L}(1) \tag{3.21}$$

$$\mathcal{L}[(u_{1_x}(x, t))] = (6x^2) \frac{1}{s^2} \tag{3.22}$$

$$[(u_{1_x}(x, t))] = (6x^2) \mathcal{L}^{-1} \left( \frac{1}{s^2} \right) \tag{3.23}$$

$$u_{1_x}(x, t) = (6x^2)(t) \tag{3.24}$$

After integrating (3.24), we get

$$u_1(x, t) = 2x^3 t \tag{3.25}$$

Now

$$P_1(u) = u_1 u_{0_{xx}} + u_0 u_{1_{xx}} + 2(u_{0_x})(u_{1_x}) \tag{3.26}$$

$$P_1(u) = (2x^3t)(2) + (x^2)(6x^2t) + 2(2x)(6x^2t) \tag{3.27}$$

$$P_1(u) = 40x^3t \tag{3.28}$$

Now

$$\mathcal{L}[u_{2x}(x, t)] = \frac{1}{s} \mathcal{L}(P_1(u)) \tag{3.29}$$

$$\mathcal{L}[u_{2x}(x, t)] = \frac{1}{s} \mathcal{L}(40x^3t) = \frac{1}{s} (40x^3) \mathcal{L}(t) \tag{3.30}$$

$$u_{2x}(x, t) = (40x^3) \mathcal{L}^{-1} \left[ \frac{1}{s^3} \right] \tag{3.31}$$

$$u_{2x}(x, t) = (40x^3) \left( \frac{t^2}{2!} \right) = 20x^3t^2 \tag{3.32}$$

After integration

$$u_2(x, t) = 5x^4t^2 \tag{3.33}$$

Now

$$P_2(u) = \frac{1}{2} (2u_2u_{0xx} + 2u_1u_{1xx} + 2u_0u_{2xx} + 4u_0xu_{2x} + 2u_1x_2) \tag{3.34}$$

$$= 210x^4t^2 \tag{3.35}$$

So

$$\mathcal{L}[u_{3x}(x, t)] = \frac{1}{s} \mathcal{L}(P_2(u)) \tag{3.36}$$

$$\mathcal{L}[u_{3x}(x, t)] = \frac{420x^4}{s^4} \tag{3.37}$$

$$u_{3x}(x, t) = \mathcal{L}^{-1} \left( \frac{420x^4}{s^4} \right) \tag{3.38}$$

$$u_{3x}(x, t) = 70x^4t^3 \tag{3.39}$$

After integrating (3.39), we have

$$u_3(x, t) = 14x^5t^3 \tag{3.40}$$

In this fashion, we can find  $u_4, u_5, \dots$

The complete solution is

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \tag{3.41}$$

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$$

$$u(x, t) = x^2 + 2x^3t + 5x^4t^2 + 14x^5t^3 + \dots \tag{3.42}$$

which is the numerical solution of (3.1) by Laplace Decomposition Method and the exact solution of (3.1) [1] is  $u(x, t) = \frac{4x^2}{(1+\sqrt{1-4xt})^2}$ .

Table 1: The values of  $u(x, t)$  and error at  $t = 0.2, x = 0.1 - 0.9$

x	Exact value	Approximate value	Error
0.1	0.0104211	0.01042119	0.0000000045
0.2	0.0435607	0.04356014	0.0000006218
0.3	0.1027526	0.10274115	0.0000115000
0.4	0.1922359	0.19214213	0.0000938000
0.5	0.3175416	0.31705000	0.0000938000
0.6	0.4861218	0.4841644	0.0004916000
0.7	0.7084380	0.70194985	0.0019574000
0.8	1.0000000	0.98103624	0.0064881000
0.9	1.3856217	1.33466772	0.0189638000

The table 1 reflects that the approximate solution obtained by Laplace Decomposition Method is nearly equal to the exact solution, provided that the values of  $x$  remains in the interval  $[0.1, 0.3]$ , when  $t$  is fixed as 0.2, since  $1 - 4xt$  is positive

but for the exact solution  $x < 1.25$ . It can be observed that the accuracy of the approximate solution is more when  $0.1 \leq x \leq 0.3$  and error increases when  $0.3 < x < 1.25$ , after this both the solutions become divergent.

#### 4. Conclusions

To obtain the numerical solution of Zabolotskaya-Khokholove equation, Laplace Decomposition Method is utilized. A series solution of Z-K equation were furnished with the help of this efficient method and desired results have been detected. This covered the uncertainties and ambiguous initial conditions that arises in most of the real world physical problems. Laplace transformation and inverse Laplace transformation is applied in this method, which has made it simple and easy. For understanding, this method is practiced to physical problems and the results obtained are very similar to the exact solution.

#### REFERENCES

[1] **Ahmed, S. and Kalim**(2015). A comparative approach to the solution of the Zabolotskaya Khokhlov Equation by Iteration Methods. *Adv. in Math. Phy.*, Article ID 7909632, 8 pages.

[2] **Kumar, M., Kumar, R. and Kumar, A.** (2014). On Similarity Solutions of Zabolotskaya Khokhlov equation. *Comp. and Math. With Appl.*, (68), pp. 454 - 463.

[3] **Tajiri, M.** (2013). Similarity reduction of the Zabolotskaya Khokhlov equation with a

dissipative term. *J. of Nonlinear Math, Phy.*, 2(3), pp. 392 – 397.

[4] **Adomian, G.** (1988). A review of the decomposition method in Applied Mathematics, *J. Math. Anal.*, 135, pp. 501–544.

[5] **Adomian, G.** (1994) Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston

[6] **Khuri, S. A.**(2001) A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *J. Math. Annl. Appl.*, 4, pp. 141-155.

[7] **Khuri, S. A.** (2004) A new approach to Bratus problem, *Appl. Math. Comp.*, 147, pp. 31-136.

[8] **Handibag, S. and Karande, B. D.** (2012) Application of Laplace Decomposition Method to Solve Linear and Nonlinear Heat Equation, *Int. J. Appl. Phy. Math.*, Vol. 2, No. 5, pp.369 – 371.

[9] **Wazwaz, A. M.** (1997) A First Course in Integral Equations, World Scientific, Singapore.

[10] **Wazwaz, A. M.** (2000) A new algorithm for calculating Adomian polynomials for nonlinear operators, 111, pp. 53-69.

[11] **Adomian, G.** (1986) The decomposition method for nonlinear dynamic systems, *J. Math. Anal. Appl.*, 120, pp. 270-283.

[12] **Datta, B.,** (1987) A new approach to the wave equation an application of the decomposition method, *J. Math. Anal. Appl.*, 1 (142), pp. 6-12.

- [13] **Adomian, G.**(1986) System of nonlinear partial differential equations, *J. Math. Anal.Appl.*,1 (115), pp. 235-238.
- [14] **Adomian, G.**(1995) Solution of coupled nonlinear partial differential equations by decomposition, *Comp. Math. Appl.*, 6 (31), pp. 117-120.
- [15] **Adomian, G., Elrod, M. and Rach, R.** (1989) A new approach to boundary value equations and application to a generalization of Airy's equation, *J. Math. Annl. Appl.*, 2 (140), pp. 554-568.
- [16] **Syam, M. I. and Hamdan, A.** (2006) An efficient method for solving Bratu equations, 176, pp. 704-713.
- [17] **Yusufoglu, E. (Agadjanov),** (2006). Numerical solution of Duffing equation by the Laplace decomposition algorithm, 177, pp. 572-580.
- [18] **Kiyamaz, O.** (2009) An algorithm for solving initial value problems using Laplace Adomian decomposition method. 3, pp. 1453-1459.
- [19] **Waswas, A. M. and Mehanna, M. S.** (2010) The combined Laplace Adomian method for handling singular integral equation of heat transfer, *Int. J. nonlin. sci.*, 10, pp. 248-252.
- [20] **Hendi, F. A.** (2011) Laplace Adomian decomposition method for solving the nonlinear Volterra Integral equation with weakly kernels, *Studies in Nonlinear Sciences*, 2, pp.129-134.
- [21] **Khan, M. and Hussain, M.** (2011) Application of Laplace decomposition method on semi-infinite domain, *Numer. Algor.*, 56, pp. 211-218.
- [22] **Zabolotskaya, E. A. and Khokhlov, R. V.** (1969) Quasi- Plane waves in the Nonlinear Acoustics of Confined Beams, *Sov. Phys. Acoustics*, V.15 N 1, pp. 35-40.